RESEARCH STATEMENT

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I am working in the field of differential geometry. My research focuses on Riemannian manifolds of **positive sectional curvature**, in particular positively curved 6-manifolds with $SU(2)$ or $SO(3)$ symmetry. I obtain the Euler characteristic bound of such 6-manifolds. I also classify such manifolds up to equivariant diffeomorphism, when the action has no exceptional orbits. A preprint of my work will be available soon.

In what follows I will give some background about positively curved manifolds, followed by my results on the topology of such manifolds in dimension 6 and lastly future projects.

1. Background

1.1. **Positive Curvature.**
The relationship between geometry and topology has fascinated mathematicians throughout the history of mathematical research. In particular, certain curvature bounds can have topological implications. In this article I will address the influence of positive curvature on the topology of the underlying manifold.

The study of Riemannian manifolds of positive sectional curvature has a long history in Riemannian geometry. These manifolds have interesting geometric and topological properties. In dimension 2, the Gauss-Bonnet formula implies that a positively curved closed surface is diffeomorphic to S^2 or $\mathbb{RP}^2$. In dimension 3, Hamilton showed that a closed 3-manifold of positive Ricci curvature is diffeomorphic to a spherical space form, using his famous Ricci flow technique [Ham82]. In higher dimensions the topology of positively curved manifolds is an active area of research in differential geometry.

By Toponogov's comparison theorem, positive curvature is characterized by the property that the sum of interior angles of geodesic triangles is always greater than π. In other words, geodesic triangles are fat. Positively curved manifolds share many geometric properties with Alexandrov spaces with lower curvature bound 0, and techniques from comparison geometry are often useful in studying them.

1.2. **Classical obstructions and Grove's symmetry program.**
For closed Riemannian manifolds with positive sectional curvature, the fundamental group is always finite by the Bonnet-Myers theorem. The Synge theorem states that the fundamental group of an even dimensional positively curved manifold is either trivial or $\mathbb{Z}/2$ and that odd dimensional positively curved manifolds are orientable. For simply connected manifolds, no topological obstructions are known to distinguish positively curved manifolds from non-negatively curved manifolds. In contrast to non-negatively curved manifolds, there are very few known examples of positively curved manifolds, including CROSSes, homogeneous spaces [Wal72][BB76], biquotiens [Zil07] and one cohomogeneity one manifold in dimension 7 [GVZ11].

In the 1990's, Karsten Grove proposed what is now called "Grove's symmetry program" [Gro02], which studies positively curved manifolds in the presence of large symmetry group with the hope that one can classify them, or possibly find new examples. This was indeed achieved in [GVZ11]. Along this direction many results were obtained. I list a few examples below:

(1) Homogeneous spaces of positive sectional curvature are classified ,which consist of CROSSes and homogeneous spaces in dimension 6,7,12,13 and 24 [Wal72][BB76].
(2) Torus actions on positively curved manifolds have been extensively studied:
    (a) The Hsiang-Kleiner theorem states that positively curved manifolds with S^1-symmetry are homeomorphic to S^4, $\mathbb{CP}^2$ or $\mathbb{RP}^4$, which was improved to an equivariant diffeomorphism by Grove and Wilking [HK89][GW14]
    (b) Grove and Searle showed that if a k-dimensional torus T^k acts effectively on an n-dimensional positively curved manifold M then $k \le [\frac{n+1}{2}]$, and that M is diffeomorphic to a CROSS when $k = [\frac{n+1}{2}]$ [GS94];
    (c) Xiaochun Rong showed that a positively curved 5-manifold with T^2-symmetry is diffeomorphic to a 5-sphere [Ron02];
    (d) Burkhard Wilking showed that if a d-dimensional torus acts effectively on an n-dimensional positively curved manifold M with $d \ge \frac{n}{4} + 1$, then M is homotopy equivalent to a CROSS [Wil03].
(3) Recently Burkhard Wilking, Lee Kennard and Michael Wiemeler proved that a even-dimensional positively curved manifold with T^5-symmetry has Euler characteristic at least 2 [KWW].

I also mention two conjectures due to Hopf, which are related to the above results. The Hopf Conjecture I states that $S^2 \times S^2$ does not admit a metric of positive sectional curvature, which holds if the manifold admits S^1-symmetry, by the Hsiang-Kleiner theorem. The Hopf Conjecture II states that even dimensional positively curved manifolds have positive Euler characteristic, which holds under T^5-symmetry by the above result.

At the end of this section I mention a result on positive curvature in dimension 5. Fabio Simas showed that if the isometry group of a positively curved 5-manifold M contains $SU(2)$ or $SO(3)$, then M is equivariantly diffeomorphic to one of the following: S^5 with a linear action, or possibly $W = SU(3)/SO(3)$ with the linear SU(2)-action, or $N^l_{m,n} = (SU(2)\times S^3)/S^1$ with trivial principal isotropy group [Sim12].

2. My Project: Positive Curvature in Dimension 6

Being part of Grove's symmetry program, my research focuses on 6-dimensional positively curved manifolds admitting isometric actions by $SU(2)$ or $SO(3)$. The last condition can also be interpreted as assuming that there exists an isometric action by a non-abelian Lie group.

In the remainder, G will denote the Lie group $SU(2)$ or $SO(3)$.

During my PhD, I proved the following result:

Theorem 2.1. *Let $M = M^6$ be a 6-dimensional closed Riemannian manifold with positive sectional curvature. Assume G acts isometrically and effectively on M. Then:*
(a) The Euler characteristic $\chi(M) = 2,\ 4,\ 6$;
(b) The principal isotropy subgroup is trivial;
(c) If the G-action has no exceptional orbits, i.e. orbits with finite stabilizer groups, then M is equivariantly diffeomorphic to S^6, $SU(3)/T^2$, or S^2-bundles over S^4.

Remark 2.2. The only known closed manifolds in dimension 6 with positive sectional curvature are S^6, $\mathbb{CP}^3$, $SU(3)/T^2$, $SU(3)//T^2$, whose Euler characteristics match the values in part (a). Regarding part (c), if there are exceptional orbits, I showed that their stabilizer groups are cyclic or dihedral groups.

I study the topology of such 6-manifolds by studying the orbit space M/G, and obtained the following result:

Theorem 2.3. *M/G is homeomorphic to either a 3-sphere or a 3-disk. Moreover,*

(a) If M/G is a 3-sphere, then there are at most 3 isolated singular orbits.
(b) If M/G is a 3-ball, then either M is diffeomorphic to S^6 or $\mathbb{CP}^3$, or M/G has at most 1 interior singular orbit and the boundary face has $SO(2)$ isotropy. Moreover in this case, either the boundary $\partial(M/G)$ has only one orbit type, or $\partial(M/G)$ has exactly 2 singular points.

I apply methods in positive curvature including Wilking's connectedness lemma ([Wil06], Theorem 1.2), the Isotropy Lemma([Wil06]), the fixed point technique and extent argument ([Gro02]) to obtain the above constraints for the orbit structure. If $G = SU(2)$ and there are no exceptional orbits, then as a special case of part (c) of Theorem 2.1, we have:

Theorem 2.4. *Let M be a positively curved 6-manifold. Assume that $G = SU(2)$ acts on M isometrically and effectively.*

(1) If the fixed point set M^G is non-empty, then M is equivariantly diffeomorphic to a linear action on S^6 or $\mathbb{CP}^3$.
(2) If M^G is empty and the action has no exceptional orbits, then the orbit space M/G is a 3-sphere with 1, 2, or 3 singular orbits. Moreover:
   (a) If M/G has 1 singular orbit, then M is equivariantly diffeomorphic to a 6-sphere with a linear action;
   (b) If M/G has 2 singular orbits, then M is equivariantly diffeomorphic to $S^2 \times S^4$ with a linear action;
   (c) If M/G has 3 singular orbits, then M is equivariantly diffeomorphic to $SU(3)/T^2$ with the $SU(2)$-action by left multiplication.

The strategy for proving the above theorem is to decompose the manifold into two parts: a neighborhood around the singular orbit and a neighborhood around the principal orbit. Then we identify their common boundaries and glue the two parts to get the original manifold.

If $G = SO(3)$ and there are no exceptional orbits, as a special case of part (c) of Theorem 2.1, we have

Theorem 2.5. *Assume $G = SO(3)$, $M/G = B^3$, and that there are no exceptional orbits. Then M is equivariantly diffeomorphic to S^6 with a linear action.*

3. Future Projects

3.1. **Continuation on 6-manifolds.** My ultimate goal is to classify all diffeomorphsim types of positively curved 6-manifolds invariant under $SU(2)$ or $SO(3)$, hopefully reducing to the 4 examples mentioned in Remark 2.2. I also aim at a more detailed description of the orbit space, listing all possible orbit structures. One main difficulty is to understand exceptional orbits. I showed that exceptional orbit strata form a 1-dimensional graph in M/G, which has interesting structures.

3.2. **Positive Curvature in Other Dimensions.** Besides dimension 6, I am also interested in the topology of positively curved manifolds in other dimensions. In dimension 5, I am able to rule out the possibility of $SU(3)/SO(3)$ admitting an $SU(2)$-invariant metric of positive sectional curvature, which was a candidate for positive curvature in Fabio's paper [Sim12] mentioned at the end of Section 1.2. I also aim at excluding other candidates in Fabio's paper, i.e. Fabios's examples $N^l_{m,n} = (SU(2) \times S^3)/S^1$ which are known to be diffeomorphic to $S^3 \times S^2$ or the unique non-trivial S^3-bundle over S^2, thus obtain a full classification in dimension 5.

In dimensions above 6, the structure of positively curved manifolds with non-Abelian symmetry is also interesting. One difficulty is that, when the cohomogeneity of the action is greater than or equal to 4, the orbit space is not necessarily homeomorphic to a manifold (possibly with boundary), which makes the orbit structure more complicated.

3.3. **Metric Foliations.** I am also interested in metric foliations on positively curved manifolds. I hope to apply the tools of holonomy fields and dual holonomy fields to prove weaker versions of Petersen-Wilhelm's conjecture [Spe16]:

Conjecture 3.1. *(Petersen-Wilhelm) Let $F \hookrightarrow M \to B$ be a Riemannian submersion of closed manifolds such that the total space M has positive sectional curvature, then the dimension of the fiber F is less than the dimension of the base B.*

The Petersen-Wilhelm conjecture is connected to positively curved 5-manifolds with non-Abelian symmetry as mentioned in Section 3.2. For example, it implies that $S^3 \times S^2$ does not admit positively curved metrics invariant under any free $SU(2)$-action on the manifold.

It is also interesting to explore similar results in the setting of metric foliations. For example, Speranca showed that for a metric foliation $\mathfrak{F}$ on a positively curved manifold M with totally geodesic leaves, if a perturbed metric has positive sectional curvature, then the dimension of M is less than twice the codimension of $\mathfrak{F}$ [Spe16]. Since isometric group actions induce metric foliations on the manifold, the tools in metric foliations can be useful in the study of positively curved manifolds under group actions.

3.4. **Homogeneous Einstein manifolds.** Besides positive curvature, I am interested in homogeneous Einstein manifolds. Wolfgang Ziller conjectured that for any closed homogeneous manifold, there exist only finitely many homogeneous Einstein metrics up to isometry and rescaling [WZ86]. Homogeneous Einstein manifolds up to dimension 6 have been extensively studied [Jen69],[ADF96],[WZ90]. In particular, homogeneous Einstein metrics on 6-dimensional closed homogeneous manifolds are classified with the only exception being $S^3 \times S^3$ [NR03]. Recently F. Belgun, V. Cortes, A. Haupt, and D. Lindemann showed the following

Theorem 3.2. *(*[BCHL18]*) If g is a left-invariant Einstein metric on $G = SU(2) \times SU(2)$ which is invariant under a non-trivial finite subgroup $\Gamma \subset Ad(G)$ such that $\Gamma \neq \mathbb{Z}/2$, then g is homothetic to the product metric or the nearly Kähler metric.*

I hope to classify all homogeneous Einstein metrics on $S^3 \times S^3$ without additional symmetry assumption. I am also interested in homogeneous Einstein manifolds in other dimensions.

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